Journal of Approximation Theory **118**, 235–245 (2002)
doi:10.1006/jath.2002.3719

# Markov-type Inequalities for Surface Gradients of Multivariate Polynomials

András Kroó[1]

*Alfréd Rényi Mathematical Institute of the Hungarian Academy of Sciences, Budapest, Reáltanoda u. 13-15 H-1053, Hungary*
E-mail: kroo@renyi.hu, kroo@math-inst.hu

*Communicated by Tamás Erdélyi*

Received August 7, 2001; accepted in revised form June 26, 2002

Let $K \subset \mathbb{R}^d$ be a compact set with a smooth boundary and consider a polynomial $p$ of total degree $\leqslant n$ such that $||p||_{C(K)} \leqslant 1$. Then we show that $|D_T p(\mathbf{x})| = o(n^2)$ for any $\mathbf{x} \in Bd\,K$ and $T$ a tangential direction at $\mathbf{x}$. Moreover, the $o(n^2)$ term is given in terms of the modulus of smoothness of $Bd\,K$. 

*Key Words:* multivariate polynomials; gradients; smooth domains..

## 1. INTRODUCTION

Let $K \subset \mathbb{R}^d$ be a compact set, and consider the set $P_n^d$ of real algebraic polynomials of $d$ variables and total degree $\leqslant n$. Denote by

$$P_n^d(K) := \{p \in P_n^d : ||p||_{C(K)} \leqslant 1\}$$

the unit ball in $P_n^d$ with respect to the uniform norm $||p||_{C(K)} := \max_{\mathbf{x} \in K} |p(\mathbf{x})|$. Furthermore, let $D_{\boldsymbol{\omega}} g$ denote the derivative of $g \in P_n^d$ in direction $\boldsymbol{\omega} \in S^{d-1}$, where $S^{d-1}$ is the Euclidian unit sphere in $\mathbb{R}^d$. Then the Markov factor of the set $K$ of order $n$ is defined as

$$M_n(K) := \max\{||D_{\boldsymbol{\omega}} p||_{C(K)} : p \in P_n^d(K),\ \boldsymbol{\omega} \in S^{d-1}\}. \tag{1}$$

It is well known that $M_n(K) \sim n^2$ for convex bodies $K \subset \mathbb{R}^d$, moreover even sharp constants are established for convex bodies. (In univariate case this dates back to Markov [9], for multivariate convex bodies see [6, 12, 13, 15].) It is also known that for cuspidal domains the Markov factors are generally

[1] Supported by the OTKA Grant T034531.

of larger (but subexponential) magnitude (see [1, 5, 7, 8, 10, 14] for details). On the other hand, the classical Bernstein Inequality for trigonometric polynomials yields that if $p \in P_n^d(S^{d-1})$, $\mathbf{x} \in S^{d-1}$, and $\boldsymbol{\omega}$ is a tangential direction to $S^{d-1}$ at $x$ then

$$|D_{\boldsymbol{\omega}} p(\mathbf{x})| \leqslant n.$$

This simple example shows that we might expect an improvement when only *tangential* derivatives at the boundary points are considered. Let us introduce the corresponding modification of the Markov factor (1). We shall consider compact sets $K \subset \mathbb{R}^d$ with $C^1$-smooth boundary (as usual $Bd\,K$ and $Int\,K$ stand for the boundary and interior of $K$, respectively). We shall say that a compact set $K \subset \mathbb{R}^d$ $(Int\,K \neq \emptyset)$ has a $C^1$ boundary if there exist an open set $D \supset Bd\,K$ and a representation function $f_K \in C^1(D)$ such that $f_K(\mathbf{x}) = 1$ for $\mathbf{x} \in Bd\,K$; $f_K(\mathbf{x}) < 1$ if $\mathbf{x} \in Int\,K \cap D$ and $\partial f_K(\mathbf{x}) \neq 0$ for $x \in Bd\,K$, where $\partial f_K$ is the gradient of $f_K$.

In the special case, when $K$ is convex, the natural choice for $f_K$ is the so-called Minkowski functional (see the examples and remarks following Theorem 1).

Clearly, $\partial f_K(\mathbf{x})$ provides the outer normal direction to $Bd\,K$ at $\mathbf{x} \in Bd\,K$. Hence, the *tangential* directions $\boldsymbol{\omega} \in S^{d-1}$ at $\mathbf{x} \in Bd\,K$ satisfy $\boldsymbol{\omega} \perp \partial f_K(\mathbf{x})$.

Now the *tangential* Markov factor of a $C^1$-domain can be introduced as

$$M_n^T(K) := \max\{|D_{\boldsymbol{\omega}} p(\mathbf{x})| : \ p \in P_n^d(K), \ \mathbf{x} \in Bd\,K, \ \boldsymbol{\omega} \perp \partial f_K(\mathbf{x})\}. \quad (2)$$

The essential difference between (1) and (2) consists in the fact that instead of all directions $\boldsymbol{\omega} \in S^{d-1}$ only tangential directions $\boldsymbol{\omega} \perp \partial f_K(\mathbf{x})$ at $\mathbf{x} \in Bd\,K$ are considered. This modification will yield a substantial improvement in the rate of Markov factors. Let

$$\omega(\partial f_K, t) := \sup\{|\partial f_K(\mathbf{x}_1) - \partial f_K(\mathbf{x}_2)| : \ \mathbf{x}_1, \mathbf{x}_2 \in D, \ |\mathbf{x}_1 - \mathbf{x}_2| \leqslant t\}$$

be the modulus of continuity of $\partial f_K$ on $D$, where $K$, $f_K$, and $D$ are as above.

This function satisfies the usual properties of moduli of continuity, e.g.,

$$\omega(\partial f_K, ct) \leqslant (1 + c)\omega(\partial f_K, t), \qquad t, c > 0.$$

Moreover, denote by $\omega_K(t)$ the *modulus of smoothness* of $Bd\,K$

$$\omega_K(t) := t\omega(\partial f_K, t). \quad (3)$$

Then we have the following:

THEOREM 1. *Let $K \subset \mathbb{R}^d$, $Int\, K \neq \emptyset$, be a compact set with a $C^1$ boundary. Then*

$$M_n^T(K) \leqslant \frac{c}{\omega_K^{-1}\left(\frac{1}{n^2}\right)}, \qquad n \in \mathbb{N} \tag{4}$$

*with some constant $c > 0$ depending only on $K$ and $d$.*

Note that $\omega_K(t) = o(t)$ for a $C^1$-domain, i.e., estimate (4) yields that $M_n^T(K) = o(n^2)$. Recalling that $M_n(K) \sim n^2$ even for balls in $\mathbb{R}^d$ we can see that tangential Markov factors have smaller magnitudes. In particular, if $\omega_K(t) \sim t^p$, $1 < p \leqslant 2$ (this is the case when, for instance, $K$ is an $\ell_p$-ball) we have $M_n^T(K) = O(n^{2/p})$. We shall also see further that (4) is sharp, in general.

In the special case when $K \subset \mathbb{R}^d$ is a *convex body*, the $C^1$-smoothness of the boundary holds whenever $K$ is *regular*, i.e., there is a *unique* supporting hyperplane to $K$ at every $\mathbf{x} \in Bd\, K$ (see [3, p. 449]). In this case for any fixed $\mathbf{x}_0 \in Int\, K$ the corresponding Minkowski functional

$$f_K(\mathbf{x}) := \inf\left\{\alpha > 0 \colon \frac{\mathbf{x} - \mathbf{x}_0}{\alpha} \in K\right\} \tag{5}$$

can be used in (3). This yields.

COROLLARY 1. *Let $K \subset \mathbb{R}^d$ be a regular convex body with $f_K$ given by* (5). *Then* (4) *holds with some $c > 0$ depending only on $K$ and $d$.*

Thus, in particular, $M_n^T(K) = o(n^2)$ whenever $K$ is a regular convex body. (This later statement was also obtained by Revesz [11].)

As a model of regular convex bodies consider the so-called "$\varphi$-ball" defined as

$$B_\varphi := \left\{\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d \colon \sum_{j=1}^{d} \varphi(|x_j|) \leqslant 1\right\},$$

where $\varphi \in C^2(0,1]$ is a strictly increasing convex continuous function on $[0,1]$, $\varphi(0) = 0$, $\varphi(1) = 1$. Assuming, in addition, that $\varphi(t)/t^2$ is a decreasing function on $(0,1]$ it is easy to show that for $K = B_\varphi$, we have $\omega_K(t) \sim \varphi(t)$. Moreover, we have the following converse to (4) which shows that Theorem 1 provides in general the best bound possible.

THEOREM 2. *There exists* $c_1 > 0$ *so that*

$$M_n^T(B_\varphi) \geqslant \frac{c_1}{\varphi^{-1}(1/n^2)}. \tag{6}$$

The quantity $M_n^T(K)$ provides bounds for the tangential derivatives of $p$ under the normalization $||p||_{C(K)} \leqslant 1$. This raises the natural question if similar results can hold under a weaker assumption $||p||_{C(Bd\,K)} \leqslant 1$? A question of this type was investigated by Bos *et al*. [2] where it was shown that on smooth algebraic curves in $\mathbb{R}^2$ the tangential Markov factor is of the size $n$ or $n^2$ depending on whether the curve is closed or not.

In [2] the authors also give an example of a smooth *nonclosed* curve in $\mathbb{R}^2$ on which Markov inequality fails to hold with any factor $n^r$, $\forall r > 0$. We shall use some ideas from [2] to modify this example for closed smooth surfaces showing that the normalization $||p||_{C(Bd\,K)} \leqslant 1$ does not lead to any meaningful estimates for $|D_{\boldsymbol{\omega}} p(\mathbf{x})|$ when $\mathbf{x} \in Bd\,K$ and $\boldsymbol{\omega} \perp \partial f_K(\mathbf{x})$. Thus, we introduce the quantity $\tilde{M}_n(K)$ defined similarly to $M_n^T(K)$ in (2) except that condition $p \in P_n^d(K)$ (i.e. $||p||_{C(K)} \leqslant 1$) is replaced by $||p||_{C(Bd\,K)} \leqslant 1$, $p \in P_n^d$. Then we have

THEOREM 3. *For any sequence of numbers* $\gamma_n \uparrow \infty$ *there exists a regular convex body* $K \subset \mathbb{R}^d$ *such that*

$$\limsup_{n\to\infty} \frac{\tilde{M}_n(K)}{\gamma_n} > 0. \tag{7}$$

Markov-type inequalities which measure the rate of growth of polynomials have numerous applications, for instance, in inverse theorems of approximation theory, comparison of various norms of polynomials, etc. Estimate (4) which exhibits an improvement of the order of tangential Markov factors on smooth surfaces yields corresponding improvements in applications mentioned above.

## 2. PROOFS

First, we shall need several auxiliary geometric lemmas which are needed for the proof of Theorem 1. From now on we assume that $K \subset \mathbb{R}^d$ has a $C^1$-boundary as defined in Section 1, and $f_K \in C^1(D)$ is the corresponding representation function with modulus of smoothness $\boldsymbol{\omega}_K$ given by (3) ($D \supset Bd\,K$ is open). Furthermore, we shall denote by $c_j, a_j, b_j$ constants depending only on $K$ and $d$.

LEMMA 1. *There exist* $c_1, c_2 > 0$ *such that whenever* $\mathbf{x} \in Bd\,K$, *for any* $\mathbf{y} \in D$ *with* $|\mathbf{x}-\mathbf{y}| \leqslant c_1$ *we have*

$$|f_K(\mathbf{x}) - f_K(\mathbf{y}) - \langle \partial f_K(\mathbf{x}), \mathbf{x}-\mathbf{y}\rangle| \leqslant c_2 \omega_K(|\mathbf{x}-\mathbf{y}|). \tag{8}$$

*Proof.* Let $c_1$ be such that $\mathbf{y} \in D$ if $|\mathbf{x}-\mathbf{y}| \leqslant c_1$. Clearly for some $\boldsymbol{\xi}_j \in D$ with $|\boldsymbol{\xi}_j - \mathbf{x}| \leqslant |\mathbf{y}-\mathbf{x}|$ $(1 \leqslant j \leqslant d)$

$$f_K(\mathbf{x}) - f_K(\mathbf{y}) = \sum_{j=1}^{d} \frac{\partial f_K}{\partial x_j}(\boldsymbol{\xi}_j)(x_j - y_j),$$

where $\mathbf{x} = (x_1, \ldots, x_d), \mathbf{y} = (y_1, \ldots, y_d)$. Thus,

$$\begin{aligned} |f_K(\mathbf{x}) - f_K(\mathbf{y}) - \langle \partial f_K(\mathbf{x}), \mathbf{x}-\mathbf{y}\rangle| &= |\sum_{j=1}^{d} (\tfrac{\partial f_K}{\partial x_j}(\boldsymbol{\xi}_j) - \tfrac{\partial f_K}{\partial x_j}(\mathbf{x}))(x_j - y_j)| \\ &\leqslant c_2 |\mathbf{x}-\mathbf{y}| \omega(\partial f_K, |\mathbf{x}-\mathbf{y}|) \\ &= c_2 \omega_K(|\mathbf{x}-\mathbf{y}|). \quad \blacksquare \end{aligned}$$

In what follows, $[\boldsymbol{a}, \boldsymbol{b}] = \{\boldsymbol{a}t + (1-t)\boldsymbol{b},\ 0 \leqslant t \leqslant 1\},\ \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$.

LEMMA 2. *There exists* $c_0 > 0$ *such that for every* $\mathbf{x} \in Bd\,K$ *we have*

$$[\mathbf{x}, \mathbf{x} - c_0 \partial f_K(\mathbf{x})] \subset Int\,K. \tag{9}$$

*Proof.* Let $c_1$ be such that $\mathbf{y} \in D$ for $|\mathbf{x}-\mathbf{y}| \leqslant c_1$ and relation (8) holds. Then setting $\mathbf{y} := \mathbf{x} - c_1 t \partial f_K(\mathbf{x}),\ 0 \leqslant t \leqslant 1$, we have by (8) using that $f_K(\mathbf{x}) = 1$

$$\begin{aligned} f_K(\mathbf{y}) &\leqslant f_K(\mathbf{x}) - \langle \partial f_K(\mathbf{x}), \mathbf{x}-\mathbf{y}\rangle + c_2 |\mathbf{x}-\mathbf{y}| \omega(\partial f_K, |\mathbf{x}-\mathbf{y}|) \\ &= 1 - c_1 t |\partial f_K(\mathbf{x})|^2 + c_3 t |\partial f_K(\mathbf{x})| \omega(\partial f_K, c_1 t |\partial f_K(\mathbf{x})|). \end{aligned} \tag{10}$$

It remains to recall now that $\partial f_K \neq 0$ on $Bd\,K$, i.e.,

$$\eta := \inf_{\tilde{\mathbf{x}} \in Bd\,K} |\partial f_K(\tilde{\mathbf{x}})| > 0. \tag{11}$$

This and (10) yield now that

$$f_K(\mathbf{y}) \leqslant 1 - c_1 \eta t + ct\omega(\partial f_K, t) < 1$$

if $0 < t < t_0$, where $t_0$ depends only on $K$. $\blacksquare$

For $\mathbf{x} \in Bd\,K$ and $\mathbf{y} \in \mathbb{R}^d$ denote by $\mathbf{y_x}$ the orthogonal projection of $\mathbf{y}$ to the hyperplane $H(\mathbf{x}) := \{\mathbf{z} \in \mathbb{R}^d\colon \langle \mathbf{z}-\mathbf{x}, \partial f_K(\mathbf{x})\rangle = 0\}$.

LEMMA 3. *There exists* $c > 0$ *such that whenever* $\mathbf{x} \in Bd\,K$, *for every* $\mathbf{y} \in Bd\,K$ *such that* $|\mathbf{x}-\mathbf{y}| \leqslant c$ *we have* $|\mathbf{x}-\mathbf{y}| \leqslant 2|\mathbf{x}-\mathbf{y_x}|$.

*Proof.* Since $f_K(\mathbf{x}) = f_K(\mathbf{y}) = 1$ we have by Lemma 1

$$|\langle \partial f_K(\mathbf{x}), \mathbf{x} - \mathbf{y} \rangle| \leqslant c_2 |\mathbf{x} - \mathbf{y}| \omega(\partial f_K, |\mathbf{x} - \mathbf{y}|). \tag{12}$$

Furthermore,

$$|\langle \partial f_K(\mathbf{x}), \mathbf{x} - \mathbf{y} \rangle| = |\partial f_K(\mathbf{x})| \, |\mathbf{x} - \mathbf{y}| \, |\cos \gamma| \tag{13}$$

with $0 \leqslant \gamma \leqslant \pi$ being the angle between $\partial f_K(\mathbf{x})$ and $\mathbf{x} - \mathbf{y}$. Thus by (11)–(13)

$$|\cos \gamma| \leqslant \frac{c_2}{\eta} \omega(\partial f_K, |\mathbf{x} - \mathbf{y}|),$$

i.e., $|\cos \gamma| \leqslant \frac{1}{2}$ when $|\mathbf{x} - \mathbf{y}| \leqslant c_6$ with $c_6$ chosen sufficiently small. Clearly, in this case

$$|\mathbf{x} - \mathbf{y}| = \frac{|\mathbf{x} - \mathbf{y}_\mathbf{x}|}{|\sin \gamma|} \leqslant 2|\mathbf{x} - \mathbf{y}_\mathbf{x}|. \quad \blacksquare$$

LEMMA 4. *There exist* $a_1, b_1 > 0$ *such that whenever* $\mathbf{x} \in Bd\,K$ *and* $\mathbf{y} \in \mathbb{R}^d$ *satisfy*

$$|\mathbf{x} - \mathbf{y}| \leqslant a_1, \quad \langle \mathbf{x} - \mathbf{y}, \partial f_K(\mathbf{x}) \rangle \geqslant 0, \tag{14}$$

$$|\mathbf{y} - \mathbf{y}_\mathbf{x}| \geqslant b_1 \omega_K(|\mathbf{x} - \mathbf{y}_\mathbf{x}|) \tag{15}$$

*it follows that* $\mathbf{y} \in K$.

*Proof.* Let $\mathbf{y} \in \mathbb{R}^d$ satisfy (14) and (15). Set

$$\mathbf{y}_0 := \mathbf{x} - \frac{\langle \mathbf{x} - \mathbf{y}, \partial f_K(\mathbf{x}) \rangle}{|\partial f_K(\mathbf{x})|^2} \partial f_K(\mathbf{x}).$$

Clearly, $\mathbf{y} - \mathbf{y}_0 \perp \partial f_K(\mathbf{x})$. Moreover, if $|\mathbf{x} - \mathbf{y}| \leqslant a_1$ with $a_1$ sufficiently small by Lemma 2 and (14) $\mathbf{y}_0 \in Int\,K$. Assume now that $\mathbf{y} \notin K$. Then with some $0 < t < 1$ we have that $\mathbf{y}^* := t\mathbf{y} + (1 - t)\mathbf{y}_0 \in Bd\,K$. As above denote by $\mathbf{y}^*_\mathbf{x}$ the orthogonal projection of $\mathbf{y}^*$ to $H(\mathbf{x})$. Then by Lemma 1 applied to $\mathbf{x}, \mathbf{y}^* \in Bd\,K$, and (11)

$$|\mathbf{y} - \mathbf{y}_\mathbf{x}| = |\mathbf{y}^* - \mathbf{y}^*_\mathbf{x}| = \frac{|\langle \partial f_K(\mathbf{x}), \mathbf{y}^* - \mathbf{x} \rangle|}{|\partial f_K(\mathbf{x})|} \leqslant \frac{c_2}{\eta} \omega_K(|\mathbf{x} - \mathbf{y}^*|).$$

In addition, applying Lemma 3 we have $|\mathbf{y}^* - \mathbf{x}| \leqslant 2|\mathbf{y}^*_\mathbf{x} - \mathbf{x}| \leqslant 2|\mathbf{y}_\mathbf{x} - \mathbf{x}|$ if $a_1 \leqslant c$. Using this in the last estimate yields with a proper $b_1 > 0$

$$|\mathbf{y} - \mathbf{y}_\mathbf{x}| < b_1 \omega_K(|\mathbf{y}_\mathbf{x} - \mathbf{x}|).$$

Thus (15) fails with this $b_1$, yielding that $\mathbf{y} \in K$ under conditions (14) and (15). ∎

*Proof of Theorem* 1. Let $p \in P_n^d(K)$, (i.e., $||p||_{C(K)} \leqslant 1), \mathbf{x} \in Bd\, K$. Without loss of generality, we may assume that $\mathbf{x} = \mathbf{0}$ and $\partial f_k(\mathbf{x}) = \partial f_k(\mathbf{0}) = (-1, 0, \ldots, 0)$. Now we need to estimate $D_{\boldsymbol{\omega}} p(\mathbf{0})$ for arbitrary $\boldsymbol{\omega} \perp \partial f_K(\mathbf{0})$. Again, it can be assumed that $\boldsymbol{\omega} = (0, 1, 0, \ldots, 0)$. (The above assumptions will hold after proper shifts and rotations.) Now it suffices to consider the bivariate polynomial

$$\tilde{p}(x, y) := p(x, y, 0, \ldots, 0) \in P_n^2$$

which evidently satisfies $|\tilde{p}(x, y)| \leqslant 1$ when $(x, y) \in \tilde{K} := \{(x, y) \in \mathbb{R}^2\colon (x, y, 0, \ldots, 0) \in K\}$. Note that $D_{\boldsymbol{\omega}} p(\mathbf{0}) = \frac{\partial \tilde{p}}{\partial y}(\mathbf{0})$. Now we shall apply Lemma 4 which yields, in particular, that whenever $(x, y) \in \mathbb{R}^2,\ x^2 + y^2 \leqslant a_1,\ x \geqslant 0$ (14), and $x \geqslant b_1 \omega_K(|y|)$ (15) we have $(x, y) \in \tilde{K}$. In other words with a suitable $a_2 > 0$ and $b_2 := b_1 \omega_K(a_2)$

$$|\tilde{p}(x, y)| \leqslant 1 \text{ if } |y| \leqslant a_2 \qquad \text{and} \qquad b_1 \omega_K(|y|) \leqslant x \leqslant b_2. \tag{16}$$

Now for arbitrary $0 < \varepsilon < a_2$ consider the ellipse

$$E_\varepsilon := \left\{ (x, y) \in \mathbb{R}^2\colon\ x = \frac{b_2}{2}(1 - \cos t),\ \ y = \varepsilon \sin t,\ \ |t| \leqslant \pi \right\}.$$

Note that $\omega_K(t)/t^2 = \omega(\partial f_K, t)/t$, and by a well-known property of moduli of continuity (applied to $\omega(\partial f_K, t)$) it can be assumed that $\omega_K(t)/t^2$ is nonincreasing. Clearly, $|y| \leqslant a_2$ and $0 \leqslant x \leqslant b_2$ for $(x, y) \in E_\varepsilon$. Therefore (16) holds for those $(x, y) \in E_\varepsilon$ which satisfy the additional condition $x \geqslant b_1 \omega_K(|y|)$. Evidently,

$$x = \frac{b_2}{2}(1 - \cos t) \geqslant \frac{b_2}{\pi^2} t^2, \qquad |y| = \varepsilon |\sin t| \leqslant \varepsilon |t| \tag{17}$$

whenever $|t| \leqslant \pi ((x, y) \in E_\varepsilon)$. Now set

$$\gamma_n := \pi \sqrt{\frac{b_1}{b_2}} \frac{1}{n}, \qquad \varepsilon := \varepsilon_n = \frac{1}{\gamma_n} \omega_K^{-1}\left(\frac{1}{n^2}\right). \tag{18}$$

We claim that whenever $x = \frac{b_2}{2}(1 - \cos t),\ \ y = \varepsilon_n \sin t$ (i.e., $(x, y) \in E_{\varepsilon_n}$), and $\gamma_n \leqslant |t| \leqslant \pi$ it follows that $x \geqslant b_1 \omega_K(|y|)$. Indeed by (17) using that $\omega_K(t)/t^2$ is nonincreasing we have

$$b_1 \omega_K(|y|) \leqslant b_1 \omega_K(\varepsilon_n |t|) \leqslant b_1 t^2 \omega_K(\varepsilon_n \gamma_n)/\gamma_n^2 = \frac{b_1 t^2}{n^2 \gamma_n^2} = \frac{b_2 t^2}{\pi^2} \leqslant x.$$

Thus, whenever $x = \frac{b_2}{2}(1-\cos t)$, $y = \varepsilon_n \sin t$ and $\gamma_n \leqslant |t| \leqslant \pi$ we have that (16) holds. Set now

$$g_n(t) := \tilde{p}\left(\frac{b_2}{2}(1-\cos t), \varepsilon_n \sin t\right), \tag{19}$$

whereby the above observation $|g_n(t)| \leqslant 1$ for every $\gamma_n \leqslant |t| \leqslant \pi$. Note that $\gamma_n = c/n$ and $g_n(t)$ is a univariate trigonometric polynomial of $\deg \leqslant n$.

Then by a Remez-type inequality for trigonometric polynomials proved by Erdélyi [4]

$$\max_{|t| \leqslant \pi} |g_n(t)| \leqslant b_5.$$

Hence using the Bernstein inequality for trigonometric polynomials

$$|g_n'(0)| \leqslant b_5 n.$$

This, (17) and (19) yield

$$|D_{\boldsymbol{\omega}} p(\mathbf{0})| = \left|\frac{\partial \tilde{p}}{\partial y}(\mathbf{0})\right| = \frac{1}{\varepsilon_n}|g_n'(0)| \leqslant \frac{b_5 n}{\varepsilon_n} = \frac{b_6}{\omega_K^{-1}(1/n^2)}.$$

This completes the proof of Theorem 1. ∎

In order to prove Theorem 2, we shall need the following proposition verified in [7].

Proposition. *Let $n \in \mathbb{N}$, $\psi \in C[0,1]$, $\psi(0) = 0$, be an increasing function of polynomial growth (i.e., $x^{-\beta}\psi(x)$ is decreasing for some $\beta > 0$). Then there exists a $p^* \in P_n^1$ such that*

$$\max_{0 \leqslant x \leqslant 1} \psi(x)|p^*(x)| \leqslant 1 \qquad \textit{and} \qquad |p^*(0)| \geqslant \frac{C}{\psi(1/n^2)}. \tag{20}$$

*Proof of Theorem* 2. Consider the $\varphi$-ball $B_\varphi$, where $\varphi \in C^2(0,1]$ is convex, increasing and $\varphi(t)/t^2$ is nonincreasing ($\varphi(0) = 0$). Then with $K = B_\varphi$, $f_K(\mathbf{x}) = \sum_{j=1}^d \varphi(|x_j|)$ ($\mathbf{x} = (x_1, \ldots, x_d)$) we have $\omega_K(t) \sim \varphi(t)$. Set $\psi := \varphi^{-1}$ in the above proposition and let $p^* \in P_n^1$, $n \in \mathbb{N}$ satisfy (20) with $\psi = \varphi^{-1}$. Set

$$p(\mathbf{x}) := x_2 p^*(1 - x_1^2), \qquad \mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d. \tag{21}$$

Then whenever $\mathbf{x} \in B_\varphi$ we have $\varphi(|x_1|) + \varphi(|x_2|) \leqslant 1$, i.e., $|x_1| \leqslant 1$ and

$$\varphi(|x_2|) \leqslant 1 - \varphi(|x_1|) \leqslant c(1 - x_1^2).$$

Hence, it easily follows that

$$|x_2| \leqslant \varphi^{-1}(c(1-x_1^2)) \leqslant c_1\psi(1-x_1^2).$$

Thus by (20) and (21)

$$|p(\mathbf{x})| \leqslant c_1\psi(1-x_1^2)|p^*(1-x_1^2)| \leqslant c_1. \tag{22}$$

On the other hand, $\mathbf{w} := (0,1,0,\dots,0)$ is a tangent direction to $B_\varphi$ at $\mathbf{x} := (1,0,\dots,0) \in Bd\, B_\varphi$. Moreover, by (20) and (21)

$$|D_{\mathbf{w}}p(\mathbf{x})| = |p^*(0)| \geqslant \frac{c}{\psi(1/n^2)} = \frac{c}{\varphi^{-1}(1/n^2)}.$$

This together with (22) verifies estimate (6). ∎

*Proof of Theorem* 3. Clearly it suffices to verify the statement of the Theorem for 2 variables, i.e. we may set $d=2$. Let $\gamma_n \uparrow \infty$ be an arbitrary sequence of positive numbers. Then we can choose a sequence of integers $n_k \in \mathbb{N}$, so that

$$n_1 = 1, \quad n_{k+1} > \gamma_{2n_k}^2, \quad k = 1,2,\dots\ . \tag{23}$$

Furthermore, set $\xi_k := a2^{-n_k},\ k=1,2,\dots$ where $a = (\sum_{k=1}^{\infty} 2^{-n_k})^{-1}$. Then, evidently,

$$\sum_{k=1}^{\infty} \xi_k = 1, \qquad \sum_{j=k+1}^{\infty} \xi_j \leqslant 2\xi_{k+1}, \quad k \in \mathbb{N}. \tag{24}$$

Finally, consider the function

$$f(x) := \sum_{k=1}^{\infty} \xi_k x^{2n_k}, \quad x \in [-1,1]. \tag{25}$$

Then $f$ is an even analytic function on $[-1,1]$, $f(0)=0$ and $f(1)=1$. Moreover, setting

$$K := \{(x,y) \in \mathbb{R}^2 \colon f(x) + y^2 \leqslant 1\}$$

it can be easily shown that $K$ is a regular convex body in $\mathbb{R}^2$.

Consider now the polynomial

$$g_k(x,y) := 1 - p_k(x) - y^2 \in P_{2n_k}^2,$$

where

$$p_k(x) := \sum_{i=1}^{k} \xi_i x^{2n_i} \in P^1_{2n_k}.$$

Then for every $(x, y) \in Bd\,K = \{(x, y) \in \mathbb{R}^2 : f(x) + y^2 = 1\}$ we have $|x| \leqslant 1$, and by (25) and (24)

$$\begin{aligned} |g_k(x, y)| &= |1 - y^2 - p_k(x)| = |f(x) - p_k(x)| \\ &\leqslant \left| \sum_{i=k+1}^{\infty} \xi_i x^{2n_i} \right| \leqslant \sum_{i=k+1}^{\infty} \xi_i \leqslant 2\xi_{k+1}. \end{aligned}$$

Hence for every $k \in \mathbb{N}$

$$||g_k||_{C(Bd\,K)} \leqslant 2\xi_{k+1}. \tag{26}$$

Clearly, at any $(x, y) \in Bd\,K$ the tangent directions to $Bd\,K$ are given by

$$\boldsymbol{\omega} := \pm \frac{1}{\sqrt{4y^2 + |f'(x)|^2}} (2y, -f'(x)) \in S^1.$$

Therefore

$$|D_{\boldsymbol{\omega}} g_k| = |\langle \partial g_k, \boldsymbol{\omega} \rangle| = \frac{2|y||f'(x) - p'_k(x)|}{\sqrt{4y^2 + |f'(x)|^2}}, \qquad (x, y) \in Bd\,K.$$

Moreover, $|y| = \sqrt{1 - f(x)}$ on $Bd\,K$, and $\sqrt{4y^2 + |f'(x)|^2} \leqslant c_1$ on $K$. Thus using the above relation

$$|D_{\boldsymbol{\omega}} g_k(x, y)| \geqslant c_2 \sqrt{1 - f(x)} |f'(x) - p'_k(x)|, \qquad (x, y) \in Bd\,K. \tag{27}$$

Obviously, with some $a > 0$

$$\sqrt{1 - f(x)} \geqslant a\sqrt{1 - x} \qquad (\tfrac{1}{2} \leqslant x \leqslant 1)$$

and for every $0 \leqslant x \leqslant 1$:

$$|f'(x) - p'_k(x)| = \sum_{i=k+1}^{\infty} \xi_i 2n_i x^{2n_i - 1} \geqslant \xi_{k+1} n_{k+1} x^{2n_{k+1} - 1}.$$

Using the last two estimates in (27) yields that for any $(x, y) \in Bd\,K$ with $\frac{1}{2} \leqslant x \leqslant 1$

$$|D_{\boldsymbol{\omega}} g_k(x, y)| \geqslant c_3 \xi_{k+1} n_{k+1} x^{2n_{k+1}-1} \sqrt{1-x}. \tag{28}$$

Now setting $x^* := \frac{2n_{k+1}-1}{2n_{k+1}}$ and $y^* := \sqrt{1 - f(x^*)}$ in (28) we have

$$|D_{\boldsymbol{\omega}} g_k(x^*, y^*)| \geqslant c_4 \xi_{k+1} \sqrt{n_{k+1}}.$$

Finally, by (23) and (26)

$$|D_{\boldsymbol{\omega}} g_k(x^*, y^*)| \geqslant c_4 \xi_{k+1} \gamma_{2n_k} \geqslant \frac{c_4}{2} \gamma_{2n_k} ||g_k||_{C(Bd\,K)},$$

where $g_k \in P^2_{2n_k}$, $(x^*, y^*) \in Bd\,K$ and $\boldsymbol{\omega}$ is a tangent direction to $Bd\,K$ at $(x^*, y^*)$. The proof of Theorem 3 is completed. ∎

## REFERENCES


1. M. Baran, Markov inequality on sets with polynomial parametrization, *Ann. Polon. Math.* **60** (1994), 69–79.
2. L. Bos, N. Levenberg, P. Milman, and B. A. Taylor, Tangential Markov inequalities characterize algebraic submanifolds of $\mathbb{R}^N$, *Indiana Univ. Math. J.* **44** (1995), 115–138.
3. N. Danford and J. T. Schwartz, "Linear Operators I," Interscience, New York, 1958.
4. T. Erdélyi, Remez-type inequalities on the size of generalized polynomials, *J. London Math. Soc.* **45** (1992), 255–264.
5. P. Goetgheluck, Inegalité de Markov dans les ensembles effilés, *J. Approx. Theory* **30** (1980), 149–154.
6. O. D. Kellogg, On bounded polynomials of several variables, *Math. Z.* **27** (1928), 55–64.
7. A. Kroó and J. Szabados, Markov–Bernstein type inequalities for multivariate polynomials on sets with cusps, *J. Approx. Theory* **102** (2000), 72–95.
8. A. Kroó, Extremal properties of multivariate polynomials on sets with analytic parametrization, *East J. Approx.* **7** (2001), 27–40.
9. A. A. Markov, Sur une question posée par D. I. Mendeleieff, *Izv. Acad. Nauk St. Petersburg* **62** (1889), 1–24.
10. W. Pawlucki and W. Plesniak, Markov's inequality and $C^\infty$ functions on sets with polynomial cusps, *Math. Ann.* **275** (1986), 467–480.
11. Sz. Révész, private communication.
12. Y. Sarantopoulos, Bounds on the derivatives of polynomials on Banach spaces, *Math. Proc. Cambridge Philos. Soc.* **110** (1991), 307–312.
13. V. I. Skalyga, Estimates of the derivatives of polynomials on convex bodies, *Proc. Steklov Inst. Math.* **218** (1997), 372–383.
14. V. Totik, On Markoff inequality, *Constr. Approx.*
15. D. R. Wilhelmsen, A Markov inequality is several dimensions, *J. Approx. Theory* **11** (1974), 216–220.